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Operator-Valued Connections, Lie Connections, and Gauge Field Theory

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Noether's first theorem tells us that the global symmetry group G_r of an action integral is a Lie group of point transformations that acts on the Cartesian product of the space-time manifold with the space of states and their derivatives. Gauge theory constructs are thus required for symmetry groups that act indiscriminately on the independent and dependent variables where the group structure can not necessarily be realized as a subgroup of the general linear group. Noting that the Lie algebra of a general symmetry group G_r can be realized as a Lie algebra g_r of Lie derivatives on an appropriately structured manifold, G_r-covariant derivatives are introduced through study of connection 1-forms that take their values in the Lie algebra g_r of Lie derivatives (operator-valued connections). This leads to a general theory of operator-valued curvature 2-forms and to the important special class of Lie connections. The latter are naturally associated with the minimal replacement and minimal coupling constructs of gauge theory when the symmetry group G_r is allowed to act locally. Lie connections give rise to the gauge fields that compensate for the local action of G_r in a natural way. All governing field equations and their integrability conditions are derived for an arbitrary finite dimensional Lie group of symmetries. The case where G_r contains the ten-parameter Poincaré group on a flat space-time M_4 is considered. The Lorentz structure of M_4 is shown to give a pseudo-Riemannian structure of signature 2 under the minimal replacement associated with the Lie connection of the local action of the Poincaré group. Field equations for the matter fields and the gauge fields are given for any system of matter fields whose action integral is invariant under the global action of the Poincaré group.

1. INTRODUCTION

The classic theory of gauge fields (Yang, 1975; Drechler and Mayer, 1977; Actor, 1979) is now recognized as an essential component in the conceptual containment of physical phenomena. It begins with the recognition of a global internal symmetry group of the salient physical state variables. This recognition is achieved by observing that the group in question leaves the action integral of the physical system invariant. Usually, although not invariably, the state space of the physical system is a representation space for the internal symmetry group, so the group action is linear. Next, the symmetry group is allowed to act locally; that is, different elements of the group act on the state variables at different points in space-time. Once local action of the group is allowed, the action integral ceases to be invariant. Restoration of invariance of the action integral is achieved by the Yang–Mills minimal replacement construct. This construct replaces ordinary partial derivatives by gauge-covariant derivatives, where the associated connection forms take their values in the matrix Lie algebra of the original linear symmetry group and compensate for the local action of the group.

There are two aspects of this construct that appear unduly restrictive. First, the group of symmetries of the action integral of a given physical system is usually much richer than just a linear internal symmetry group. It can be calculated without difficulty and its general properties are well known from the pioneering work of E. Noether (1918). In particular, the group action can occur on both the physical state variables and the space-time labels, and the group action need not be linear. All that is required is that all quantities have well-defined Lie derivatives with respect to vector fields on an appropriately structured space. The symmetry group is then obtained by exponentiation of a Lie algebra of Lie derivatives.

This brings us to the second aspect. The classic Yang–Mills minimal replacement construct introduces connection forms that take their values in the matrix Lie algebra of the linear internal symmetry group, while the general situation involves a group that is the exponentiation of a Lie algebra of Lie derivatives. It is then almost self-evident that the general case should involve connection forms that take their values in a Lie algebra of Lie derivatives; that is, we have to be able to deal with operator-valued connections. This aspect of the problem is dealt with in Sections 2–5. The problem is first analyzed in a general context and includes a full account of operator-valued curvature forms. Lie connections are then introduced for the express purpose of providing a simple and direct approach to gauge theory.

Sections 6–9 obtain a gauge theory for any group that is an invariance group for an action integral. The general case is first analyzed, including a derivation of all relevant field equations. Application is then made to situations in which the action integral is invariant under the Poincaré group. The ease with which gauging by the Poincaré group is effected is a simple and efficacious test of the general theory. The details of this construct give a possibly more direct and fundamental basis for many of the practices in the current literature (Kikkawa et al., 1983).

The paper concludes with several observations concerning important extensions and implications.

2. BASE SPACE, KINEMATIC SPACE, AND LIE GROUPS OF POINT TRANSFORMATIONS

The base space, or manifold of independent variables, is an n-dimensional differentiable manifold M_n with local coordinates $\{x^i|1 \leqslant i \leqslant n\}$. In practice, $n = 4$ and M_n will be a flat space-time manifold. The volume element of M_n is denoted by

$$\mu = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \tag{1}$$

Since $\{\partial_i = \partial/\partial x^i|1 \leqslant i \leqslant n\}$ forms a natural basis for $T(M_n)$, the quantities

$$\mu_i = \partial_i \lrcorner \mu, \qquad 1 \leqslant i \leqslant n \tag{2}$$

are well defined. They form a conjugate basis for $\Lambda^{n-1}(M_n)$ with the properties (Edelen, 1980)

$$d\mu_i = 0, \qquad dx^i \wedge \mu_j = \delta^i_j \mu \tag{3}$$

There will be a number of different spaces involved in this discussion. For simplicity, $T(W)$ will be used to denote the tangent space of W and $\Lambda(W)$ denotes the exterior algebra of differential forms over W. If $S = R \times T$, then $\Lambda(R)$ and $\Lambda(T)$ will denote the exterior algebras of differential forms over R and T, respectively. If π_1 and π_2 are the projections onto the first and second factors, respectively, $\pi_1 : R \times T \to R$, $\pi_2 : R \times T \to T$, then $\Lambda(R)$ and $\Lambda(T)$ trivially lift to subspaces $(\pi_1)^*\Lambda(R)$ and $(\pi_2)^*\Lambda(T)$ of $\Lambda(R \times T)$, respectively.

Suppose that we are given a system of N quantities on M_n. In practice, these will be the state variables of a dynamical system on space-time. For the purposes of this discussion, let $\mathbb{R}^N$ be the range space of the given N quantities. The space K is defined by

$$K = M_n \times \mathbb{R}^N \tag{4}$$

[see Edelen (1980) for this construction] and will be referred to as *kinematic* *space*. In order to simplify matters, we assume that K is referred to a system of local coordinates $\{z^A|1 \leqslant A \leqslant n+N\}$ for the time being. The general discussion will be carried out in this context. Only later, after identifying the state variables, will it be necessary to identify some of the z^A's with the x^i's.

Let G_r be a given r-parameter Lie group and let g_r be its Lie algebra. We assume that G_r acts on K as an r-parameter Lie group of point transformations,

$$`z^A = \exp(u^a V_a) z^A \tag{5}$$

where $\{u^a|1 \leqslant a \leqslant r\}$ is a system of canonical parameters for G_r and $\{V_a \in T(K)|1 \leqslant a \leqslant r\}$ is a basis for g_r in this representation. We thus have

$$[V_a, V_b] = C^e_{ab} V_e \tag{6}$$

where C^e_{ab} are the structure constants of G_r.

Since $V_a \in T(K)$, each V_a acts on the collection of C^∞ functions $\Lambda^0(K)$ by

$$V_a : \Lambda^0(K) \to \Lambda^0(K) | `f = V_a f$$

The Lie algebra g_r may thus be realized in terms of the mappings $V_a : \Lambda^0(K) \to \Lambda^0(K)$. We denote this situation by $g_r(V_a; \Lambda^0(K))$. It is then a trivial matter to see that g_r may also be realized by $g_r(£_a; \Lambda(K))$ since

$$[£_a, £_b] = £_a £_b - £_b £_a = C^e_{ab} £_e, \qquad £_a = £_{V_a} \tag{7}$$

and $\Lambda(K)$ is a domain for the Lie derivative. The r operators $\{£_a|1 \leqslant a \leqslant r\}$ then form a basis for Lie algebra $g_r(£_a; \Lambda(K))$ and the group G_r acts on $\Lambda(K)$ by

$$`\omega = \exp(u^a £_a)\omega, \qquad \omega \in \Lambda(K) \tag{8}$$

In view of these considerations, we can shift to the space

$$\mathscr{G} = G_r \times K \tag{9}$$

with local coordinates $\{u^a, z^A|1 \leqslant a \leqslant r, 1 \leqslant A \leqslant n+N\}$. It is now just one more step to consider the larger structure $\Lambda(\mathscr{G})$.

Let $\bar{d}$ denote the exterior derivative on $\mathscr{G}$. We then have

$$\bar{d} = d + d_u, \qquad d = \bar{d}\big|_K, \qquad d_u = \bar{d}\big|_{G_r} \tag{10}$$

where $\bar{d}|_R$ denotes the restriction to R, and

$$£_a u^b = 0, \qquad £_a \bar{d}u^b = 0 \tag{11}$$

because $V_a \in T(K)$ and hence $\exp(u^a £_a)$ restricted to $\Lambda(G_r)$ is the identity. It is therefore consistent to allow G_r to act on $\Lambda(\mathscr{G})$ by

$$`\Omega = \exp(u^a £_a)\Omega, \qquad \Omega \in \Lambda(\mathscr{G}) \tag{12}$$

In particular, we have $g_r(£_a; \Lambda(\mathscr{G}))$ and hence $\{£_a | 1 \leqslant a \leqslant r\}$ is an operator basis for $g_r(£_a; \Lambda(\mathscr{G}))$. It is then a simple matter to see that (12) applied to $(\pi_2)^*\Lambda(K)$ reproduces (8). The reason why we have to go to the larger structure $\Lambda(\mathscr{G})$ is because we will be interested in what happens when $\bar{d}$ is applied to the image of $\Lambda(K)$ under the action of G_r ($`\omega = \exp(u^a £_a)\omega$) in order to prepare for an eventual dependence of the u^a's on the x^i's that comes about by a mapping from M_n into G_r.

3. OPERATOR-VALUED CONNECTION 1-FORMS

For any $\omega \in \Lambda(K)$, we have $d(\exp(u^a £_a)\omega) = \exp(u^a £_a)\, d\omega$ because d and $£_a$ commute and the u's are fixed. Thus, the exterior derivative on $\Lambda(K)$ transforms covariantly under the action of G_r. When consideration is shifted to $\Lambda(\mathscr{G})$, things are no longer so simple because

$$\bar{d}(\exp(u^a £_a)\omega) = d(\exp(u^a £_a)\omega) + d_u(\exp(u^a £_a)\omega) \tag{13}$$

by (10), and d_u does not commute with $\exp(u^a £_a)$. A direct way around this difficulty is to set

$$d_u(\exp(u^a £_a)\Omega) - \exp(u^a £_a) d_u\Omega = \exp(u^a £_a)(\Gamma \wedge \Omega) - `\Gamma \wedge \exp(u^a £_a)\Omega \tag{14}$$

as suggested by the Yang–Mills construction of gauge-covariant derivatives (Yang, 1975; Drechler and Mayer, 1977). Noting that $g_r(£_a; \Lambda(\mathscr{G}))$ has the operators $\{£_a | 1 \leqslant a \leqslant r\}$ as a basis, we take Γ to be an element of $\Lambda^1(\mathscr{G})$ with values in $g_r(£_a; \Lambda(\mathscr{G}))$ and $`\Gamma$ to be the image of Γ under the action of G_r that is defined by (14). We therefore have

$$\Gamma = W^a £_a, \qquad W^a \in \Lambda^1(\mathscr{G}) \tag{15}$$

and (13) and (14) combine to give

$$\bar{d}(\exp(u^a £_a)\Omega) + {}^{`}\Gamma \wedge \exp(u^a £_a)\Omega = \exp(u^a £_a)(\bar{d}\Omega + \Gamma \wedge \Omega) \quad (16)$$

This shows that

$$D\Omega = \bar{d}\Omega + \Gamma \wedge \Omega, \qquad {}^{`}D{}^{`}\Omega = \bar{d}\,{}^{`}\Omega + {}^{`}\Gamma \wedge {}^{`}\Omega \quad (17)$$

serve to define a *G_r-covariant exterior derivative* on $\Lambda(\mathscr{G})$;

$${}^{`}D{}^{`}\Omega = \exp(u^a £_a)(D\Omega), \qquad {}^{`}\Omega = \exp(u^a £_a)\Omega \quad (18)$$

It is therefore consistent to refer to $\Gamma = W^a £_a$ as an *operator-valued connection* on $\Lambda(\mathscr{G})$.

The transformation law for the operator-valued connection Γ on $\Lambda(\mathscr{G})$ obtains directly from (14) and (15):

$${}^{`}\Gamma \wedge \exp(u^a £_a)\Omega = \exp(u^a £_a)(W^b \wedge £_b\Omega + d_u\Omega) - d_u(\exp(u^a £_a)\Omega) \quad (19)$$

For the moment, set

$$u^a £_a = £_R, \qquad R = u^a V_a \quad (20)$$

We then have (Schouten, 1954)

$$\exp(£_R)£_a \exp(-£_R)\alpha = G_a^b(u^e)£_b\alpha \quad (21)$$

for any linear geometric object field α, where the G's are functions of the u's only that are defined by

$$\exp(£_R)V_a = G_a^b(u^e)V_b \quad (22)$$

The relations (21) show that G_r acts on its Lie algebra $g_r(£_a; \Lambda(\mathscr{G}))$ by the adjoint representation since (22) shows that the G's give the adjoint representation on the vector space $\text{span}(V_a | 1 \leqslant a \leqslant r)$. If we now set $\alpha = \exp(£_R)\beta$ in (21), it follows that

$$\exp(£_R)£_a\beta = G_a^b £_b \exp(£_R)\beta \quad (23)$$

for any element β of $\Lambda(\mathscr{G})$. Next, we note that

$$\exp(£_R)(\alpha \wedge \beta) = (\exp(£_R)\alpha) \wedge \exp(£_R)\beta \quad (24)$$

for any $\alpha, \beta \in \Lambda(\mathscr{G})$ because $\exp(£_R)(\alpha \wedge \beta) = T_R^*(\alpha \wedge \beta)$ and $T_R : K \to K$ is the automorphism of K that is generated by the flow of R with canonical orbital parameter equal to unity. A combination of (23) and (24) shows that

$$\begin{aligned}\exp(£_R)\big(W^b \wedge £_b\Omega\big) &= \big(\exp(£_R)W^b\big)\wedge \exp(£_R)£_b\Omega \\ &= \big(\exp(£_R)W^b\big)\wedge G_b^a £_a \exp(£_R)\Omega\end{aligned}$$

When this is put back into (19), we have

$$\begin{aligned}`\Gamma \wedge \exp(£_R)\Omega &= \big(\exp(£_R)W^b\big)G_b^a \wedge £_a\exp(£_R)\Omega \\ &\quad + \exp(£_R)\, d_u\Omega - d_u\big(\exp(£_R)\Omega\big)\end{aligned} \tag{25}$$

The standard equations for the Lie group G_r and the fact that Ω is a differential form on $\mathscr{G}$ with coefficients from $\Lambda^0(\mathscr{G}) = \Lambda^0(G_r \times K)$ show that

$$\frac{\partial}{\partial u^a}\big(\exp(£_R)\Omega\big) = \lambda_a^b £_b \exp(£_R)\Omega + \exp(£_R)\frac{\partial}{\partial u^a}\Omega$$

where the λ's are functions of the u's only for which $(\partial/\partial u^a)`z^A = \lambda_a^b(u)\,'V_b\langle `z^A\rangle$, $`z^A = \exp(u^b V_b)z^A$. Accordingly, we obtain

$$d_u\big(\exp(£_R)\Omega\big) = \exp(£_R)\, d_u\Omega + \bar{d}u^a \wedge \lambda_a^b £_b \exp(£_R)\Omega \tag{26}$$

When (26) is substituted into (25), it follows that

$$`\Gamma \wedge \exp(£_R)\Omega = \big(G_a^b \exp(£_R)W^a - \lambda_a^b\, \bar{d}u^a\big)\wedge £_b \exp(£_R)\Omega \tag{27}$$

Since $\{£_b | 1 \leqslant b \leqslant r\}$ is a basis for $g_r(£_a;\ \Lambda(\mathscr{G}))$, this relation can be satisfied simultaneously for all $\Omega \in \Lambda(\mathscr{G})$ if and only if

$$`\Gamma = `W^b £_b \tag{28}$$

$$`W^b = G_a^b \exp(u^e £_e)W^a - \lambda_a^b\, \bar{d}u^a \tag{29}$$

Thus, *the image of any operator-valued connection $\Gamma = W^a £_a$ under the action of G_r is an operator-valued connection $`\Gamma = `W^a £_a$, and the 1-forms $W^a \in \Lambda^1(\mathscr{G})$ transform under action of G_r by the generalized gauge transformations* (29). This latter result is not unexpected, for $\Gamma = W^a £_a$ is an operator-valued connection and hence the W^a's should transform inhomogeneously under the action of G_r. The unusual aspect is the very complicated dependence, $\exp(u^e £_e)W^a$, on the original 1-forms W^a.

4. PROPERTIES OF D AND OPERATOR-VALUED CURVATURE FORMS

Starting with (15) and (17),

$$D\omega = \bar{d}\omega + W^a \wedge £_a\omega \tag{30}$$

for any $W^a \in \Lambda(\mathscr{G})$, it is easily shown that

$$D(\alpha+\beta) = D\alpha + D\beta \tag{31}$$

$$D(\alpha\wedge\beta) = (D\alpha)\wedge\beta + (-1)^{\deg(\alpha)}\alpha\wedge D\beta \tag{32}$$

The G_r-covariant exterior derivative is thus an antiderivation on $\Lambda(\mathscr{G})$. The analogy with the exterior derivative stops here, however.

Since $D\omega$ belongs to $\Lambda(\mathscr{G})$, we have

$$£_a D\omega = £_a\bar{d}\omega + £_a W^b \wedge £_b\omega + W^b \wedge £_a £_b\omega$$

On the other hand,

$$D£_a\omega = \bar{d}£_a\omega + W^b \wedge £_b £_a\omega = £_a\bar{d}\omega + W^b\wedge £_b £_a\omega$$

because $\bar{d}$ and $£_a$ commute. When these are combined, the commutator of $£_a$ and D is seen to have the evaluation

$$(£_a D - D£_a)\omega = £_a W^b \wedge £_b\omega + W^b \wedge (£_a £_b - £_b £_a)\omega$$

Thus, when (7) is used, we have

$$(£_a D - D£_a)\omega = \rho_a \wedge \omega \tag{33}$$

where

$$\rho_a = \rho_a^b £_b \tag{34}$$

is a system of operator-valued 1-forms and

$$\rho_a^b = £_a W^b + C_{ae}^b W^e \tag{35}$$

are 1-forms on $\mathscr{G}$.

If we start with an element α from $(\pi_2)^*\Lambda(K)$, $D\alpha$ belongs to $\Lambda(\mathscr{G})$ but not to $(\pi_2)^*\Lambda(K)$. It is for precisely this reason that the space $\mathscr{G}$ was introduced in the first place, for $D\alpha \in \Lambda(\mathscr{G})$ allows us to apply the operator D to this quantity again. In particular, since (17) holds, we have

$$`D`D\exp(u^a £_a)\omega = \exp(u^a £_a)(DD\omega) \tag{36}$$

and hence $DD\omega$ is G_r covariant. A direct substitution using $D = \bar{d} + W^a \wedge £_a$ gives

$$DD\omega = \Theta \wedge \omega \tag{37}$$

where

$$\Theta = \theta^a £_a, \qquad \theta^a = DW^a + \tfrac{1}{2} C^a_{be} W^b \wedge W^e \tag{38}$$

Direct analogy with the results of differential geometry (Schouten, 1954; Sternberg, 1964) and gauge theory (Drechler and Mayer, 1977; Rund, 1982) suggests that Θ be referred to as the *operator-valued curvature* 2-*form* associated with the operator-valued connection 1-form $\Gamma = W^a £_a$ and that $\{\theta^a | 1 \leqslant a \leqslant r\}$ are the *curvature* 2-*forms* on $\Lambda(\mathscr{G})$ that arise from the connection 1-forms $\{W^a | 1 \leqslant a \leqslant r\}$. This is further borne out by noting that (37) is a G_r-covariant equation and hence [see (20)]

$$\begin{aligned} `\theta^a \wedge £_a `\Omega &= \exp(£_R)(\theta^b \wedge £_b \Omega) = (\exp(£_R)\theta^b) \wedge \exp(£_R) £_b \Omega \\ &= (\exp(£_R)\theta^b) \wedge G^a_b £_a \exp(£_R)\Omega \\ &= G^a_b(\exp(£_R)\theta^b) \wedge £_a `\Omega \end{aligned}$$

when (23) is used. Accordingly, the curvature 2-forms transform under the action of G_r by the homogeneous transformation law

$$`\theta^a = G^a_b \exp(u^e £_e)\theta^b \tag{39}$$

We now look at the two expressions $D(DD\omega)$ and $DD(D\omega)$. Since

$$DD(D\omega) = \Theta \wedge D\omega = \theta^a \wedge £_a D\omega$$

and

$$D(DD\omega) = D(\theta^a \wedge £_a \omega) = D\theta^a \wedge £_a \omega + \theta^a \wedge D£_a \omega$$

we have

$$D(DD\omega) - DD(D\omega) = D\theta^a \wedge £_a \omega + \theta^a \wedge (D£_a - £_a D)\omega$$

Accordingly, when (33) is used, we obtain

$$D(DD\omega) - DD(D\omega) = \beta \wedge \omega$$

where

$$\beta = \beta^a \pounds_a, \qquad \beta^a = D\theta^a - \rho^a_e \wedge \theta^e \tag{40}$$

An elementary calculation based upon (35), (38) and the Jacobi identity shows that $\beta^a \equiv 0$ on $\mathscr{G}$ for any choice of $W^a \in \Lambda^1(\mathscr{G})$. We accordingly have the desired result,

$$D(DD\omega) \equiv DD(D\omega) \tag{41}$$

and the corresponding Bianchi identifies $\beta^a \equiv 0$; that is,

$$D\theta^a = \rho^a_e \wedge \theta^e \tag{42}$$

There is an interesting point that should be observed here. Slight rearrangements of (38) and (42) give

$$DW^a = \theta^a - \tfrac{1}{2} C^a_{be} W^b \wedge W^e, \qquad D\theta^a = \rho^a_e \wedge \theta^e \tag{43}$$

Accordingly, the ideal

$$\overline{W} = I\{W^1, \ldots, W^r, \theta^1, \ldots, \theta^r\} \tag{44}$$

of $\Lambda(\mathscr{G})$ is G_r-covariant differentially closed; that is,

$$D\overline{W} \subset \overline{W} \tag{45}$$

This observation is of particular use in certain applications associated with isovector methods (Edelen, 1980).

An exceptional aspect of the G_r-covariant exterior derivative operator D is that *any constant element of* $\Lambda^0(\mathscr{G})$ *is* G_r*-covariant constant*, as follows directly from $Dk = \bar{d}k + W^a \wedge \pounds_a k = 0$. We therefore have

$$DC^a_{be} = 0 \tag{46}$$

since the structure constants of G_r are constant functions on $\mathscr{G}$. It is useful to compare this one-line derivation of (46) for operator-valued connections with the more customary approach in which derivation of the same result usually covers pages.

The fields of 1-forms W^a that occur in the operator-valued connection $\Gamma = W^a £_a$ are often referred to in the literature as compensating fields; that is, fields that compensate for changes that arise from the local action of the group G_r. This same interpretation obtains here. To see this, suppose that ω is a G_r-invariant form, $\omega = \exp(u^a £_a)\omega$. We then have $£_a \omega = 0$ and hence $D\omega = \bar{d}\omega + W^a \wedge £_a \omega = \bar{d}\omega$. *The G_r-covariant exterior derivative of a G_r-invariant form ω reduces to the exterior derivative of ω*:

$$£_a \omega = \omega, \qquad 1 \leqslant a \leqslant r \Rightarrow D\omega = \bar{d}\omega \tag{47}$$

The G_r-covariant exterior derivative thus differs from the exterior derivative only if the action of G_r changes things. This result is a special case of a general situation that will be of importance later. Let I be an ideal of $\Lambda(\mathscr{G})$. It then follows directly from $D\omega = \bar{d}\omega + W^a \wedge £_a \Omega$ that

$$£_a \omega \equiv 0 \bmod I \Rightarrow D\omega \equiv \bar{d}\omega \bmod I \tag{48}$$

5. LIE CONNECTIONS

Up to this point, the 1-forms W^a have been arbitrary elements of $\Lambda^1(\mathscr{G})$. We now specialize to the important case where the W^a are invariant under transport along the orbits of G_r; that is,

$$£_b W^a = 0 \tag{49}$$

for all values of a and b in the range 1 through r. Operator-valued connections $\Gamma = W^a £_a$ with the W^a's satisfying (49) will be referred to as *Lie connections*.

The constraints (49) are not as severe as might appear on first reading, for W^a are 1-forms on $\mathscr{G}$ rather than on K and $£_a u^b = 0$ are identically satisfied on $\mathscr{G}$. The full scope of this can be seen by setting $W^a = T^a + w_b^a \, \bar{d}u^b$ with $£_b T^a = 0$. In this event, (49) is satisfied provided $V_b\langle w_e^a \rangle = 0$ hold. However, $V_b\langle w_e^a \rangle = 0$ is a complete system of linear first-order partial differential equations because $[V_a, V_b] = C_{ab}^e V_e$. Thus, since $\{u^a | 1 \leqslant a \leqslant r\}$ are r primitive integrals of $V_a\langle f \rangle = 0$, we have $w_e^a = \Psi_e^a(u^b; \eta^\sigma)$ where the Ψ's are arbitrary C^1 functions of their arguments and $\{\eta^\sigma | 1 \leqslant \sigma \leqslant n + N - r\}$ together $\{u^a | 1 \leqslant a \leqslant r\}$ constitute a complete system of primitive integrals of $V_a\langle f \rangle = 0$. Here, of course, it is assumed that the set $\{\eta^\sigma\}$ is vacuous if $n + N \leqslant r$.

If $\Gamma = W^a £_a$ is a Lie connection, we have

$$DW^a = \bar{d}W^a + W^b \wedge £_b W^a = \bar{d}W^a \tag{50}$$

Equations (38) then show that the associated *Lie curvature* 2-forms θ^a are given by

$$\theta^a = \bar{d}W^a + \tfrac{1}{2}C^a_{be}W^b \wedge W^e \tag{51}$$

which are the familiar representation for curvature 2-forms in gauge theory (Yang, 1975; Drechler and Mayer, 1977).

Satisfaction of (49) implies $\exp(u^e £_e)W^a = W^a$. Thus, if $\Gamma = W^a £_a$ is a Lie connection, (27) and (28) give

$$`\Gamma = `W^a £_a \tag{52}$$

$$`W^a = G^a_b W^b - \lambda^a_b \bar{d}u^b \tag{53}$$

Thus, if G_r is restricted to a constant section ($\bar{d}u^b = 0$), the W^a's transform by the adjoint representation. In the general case, it is useful to write

$$`W^a = G^a_b W^b - \lambda^a \tag{54}$$

where

$$\lambda^a = \lambda^a_b \bar{d}u^b \tag{55}$$

are 1-forms on G_r that satisfy the Maurer equations

$$\bar{d}\lambda^a = \tfrac{1}{2}C^a_{be}\lambda^b \wedge \lambda^e \tag{56}$$

Noting that G^a_b and λ^a_b are functions of the u's only, it follows that $£_m G^a_b = 0$, $£_m \lambda^a_b \bar{d}u^b = 0$. Accordingly, (5.5) gives

$$£_m `W^a = G^a_b £_m W^a = 0 \tag{57}$$

and hence *the action of G_r takes Lie connections into Lie connections*. The collection of all Lie connections is thus closed under the action of G_r.

If θ^a are the Lie curvature 2-forms of a Lie connection $\Gamma = W^a £_a$, (51) shows that

$$£_m \theta^a = \bar{d}£_m W^a + \tfrac{1}{2}C^a_{be}\left(£_m W^b \wedge W^e + W^b \wedge £_m W^e\right) = 0 \tag{58}$$

We thus have $\exp(u^e \pounds_e)\theta^a = \theta^a$, and hence (39) gives

$$`\theta^a = G^a_b \theta^b \tag{59}$$

Lie curvature 2-forms transform under action of G_r by the adjoint representation. This result lies at the heart of later matters since it provides the means whereby a G_r-invariant 4-form may be constructed. The coefficients $\{G^a_b(u)\}$ of the adjoint action of any element of G_r satisfy (Rund, 1982)

$$G^a_e(u) C^e_{br} = C^a_{fm} G^f_b(u) G^m_r(u) \tag{60}$$

and hence

$$C_{ab} = C_{fm} G^f_a(u) G^m_b(u) \tag{61}$$

where

$$C_{ab} = C^e_{am} C^m_{be} = C_{ba} \tag{62}$$

are the components of the Cartan–Killing form on G_r. It is then a simple matter to see that

$$\rho = C_{ab}\theta^a \wedge \theta^b \tag{63}$$

is a G_r-invariant 4-form on $\mathscr{G}$ for any Lie connection $\Gamma = W^a \pounds_a$ (i.e., $C_{ab}\theta^a \wedge \theta^b = C_{ab} G^a_e G^b_f \theta^e \wedge \theta^f = C_{ef}\theta^e \wedge \theta^f$). We note as a matter of consistency that (53),

$$`\theta^a = \bar{d}`W^a + \tfrac{1}{2} C^a_{be}`W^b \wedge `W^e \tag{64}$$

and the group equations (Rund, 1982)

$$\bar{d}\lambda^a = \tfrac{1}{2} C^a_{be}\lambda^b \wedge \lambda^e, \qquad \bar{d}G^a_b = C^a_{ef} G^f_b \lambda^e \tag{65}$$

lead directly to the transformation law (59). Conversely, (51), (53), (59), and (65) lead to the determination (64) for θ^a: *evaluations of θ^a in terms of W^a are G_r invariant.*

Suppose that W^a defines a Lie connection and that

$$W^a = g^a_b(\bar{u})\lambda^b_e(\bar{u})\, d\bar{u}^e, \qquad G^a_b(\bar{u}) g^b_e(\bar{u}) = \delta^a_e \tag{66}$$

for given $\{\bar{u}^a\}$. In this event, the transformation generated by $\{\bar{u}^a\}$ will annihilate the W^a's, as follows directly from (53). We then have $0 = \theta^a = G^a_b(\bar{u})\theta^b$ and hence we see that $\theta^a = 0$ in this case. Conversely, it is a lengthy but straightforward calculation to show that $\theta^a = 0$ only if the W^a's are given by (66) for some choice of the $\{\bar{u}^a\}$. The considerations combine to give the following important result. *A Lie connection can be annihilated by an appropriate choice of an element of G_r if and only if the associated Lie curvature 2-forms vanish.*

6. THE LIE GROUP OF SYMMETRIES OF AN ACTION INTEGRAL

It is assumed, for the purposes of this discussion, that a physical system in a flat, four-dimensional ($n = 4$) space-time M_4 is described by a system of m fields $\{\phi^\alpha(x^j) | 1 \leqslant \alpha \leqslant m\}$. These fields give rise to a kinematic space K of dimension $n + N = 4 + 5m$ with local coordinates $\{x^i, q^\alpha, y_i^\alpha | 1 \leqslant i \leqslant 4, 1 \leqslant \alpha \leqslant m\}$ and *contact* 1-*forms*

$$C^\alpha = dq^\alpha - y_i^\alpha \, dx^i \tag{67}$$

The $5m$-dimensional space with local coordinates $\{q^\alpha, y_i^\alpha\}$ is the range space for the ϕ's and their first partial derivatives. Realization of the actual fields obtains through the class $\mathscr{R}$ of regular maps $\Phi: M_4 \to K$ such that

$$\Phi^*\mu \neq 0 \tag{68}$$

and

$$\Phi^*C^\alpha = 0, \qquad 1 \leqslant \alpha \leqslant m \tag{69}$$

The requirement (68) forces Φ to map M_4 onto a four-dimensional section of K with a nonvanishing projection onto M_4. We may therefore assume that $\Phi^*x^i = x^i$, $1 \leqslant i \leqslant 4$ without loss of generality. Accordingly, any Φ that satisfies (68) has a realization

$$\Phi | x^i = x^i, \qquad q^\alpha = \phi^\alpha(x^j), \qquad y_i^\alpha = \phi_i^\alpha(x^j)$$

When this is used in conjunction with the conditions (69), it is easily seen that $\phi_i^\alpha(x^j) = \partial_i \phi^\alpha(x^j)$ (see Edelen, 1980, Chap. 2). Thus, any $\Phi \in \mathscr{R}$ has a realization

$$\Phi | x^i = x^i, \qquad q^\alpha = \phi^\alpha(x^j), \qquad y_i^\alpha = \partial_i \phi^\alpha(x^j) \tag{70}$$

Any regular map $\Phi: M_4 \to K$ thus defines a four-dimensional section of K that is the graph of the fields $\phi^\alpha(x^j)$ and their first partial derivatives.

The contact forms C^α give rise to the contact ideal

$$C = I\{C^1, \ldots, C^m\} \tag{71}$$

of $\Lambda(K)$. If $V \in T(K)$, we have

$$V = v^i \partial_i + v^\alpha \partial_\alpha + v_i^\alpha \partial_\alpha^i \tag{72}$$

where $\partial_i = \partial/\partial x^i$, $\partial_\alpha = \partial/\partial q^\alpha$, $\partial_\alpha^i = \partial/\partial y_i^\alpha$. The collection

$$TC(K) = \{V \in T(K) | \pounds_V C \subset C\} \tag{73}$$

constitutes the set of all *isovector fields* of the contact ideal C. Any $V \in TC(K)$ has the form (Edelen, 1980)

$$V = f^i(x^j, q^\beta)\partial_i + f^\alpha(x^j, q^\beta)\partial_\alpha + Z_i\langle V \lrcorner C^\alpha\rangle \partial_\alpha^i \tag{74}$$

with

$$Z_i = \partial_i + y_i^\alpha \partial_\alpha \tag{75}$$

Isovector fields have the property that they transport regular sections of K into regular sections of K and hence they preserve the correlation of $\Phi^* y_i^\alpha$ with $\partial_i \Phi^* q^\alpha$.

Let $L(x^j, q^\alpha, y_i^\alpha)$ be a given smooth function on K. The action integral associated with any regular map Φ is defined in terms of L by

$$A[\Phi] = \int_{M_4} \Phi^*(L\mu) = \int_{M_4} L(x^i, \phi^\alpha(x^j), \partial_i \phi^\alpha(x^j))\, dx^1\, dx^2\, dx^3\, dx^4 \tag{76}$$

Thus, the action is a map $A: \mathscr{R} \to \mathbb{R}$ of regular maps into the real line and L is the Lagrangian function for the physical system on kinematic space K.

The full content of the calculus of variations is available in this context, for the variational process is directly generated by study of the deformations of the action that arise as a consequence of transport along the orbits of isovector fields of the contact ideal C (see Edelen, 1980, Chap. 5). Let $T_V(s)$ be the one-parameter family of automorphisms of K that is generated by transport of points of K along the orbits of $V \in TC(K)$. Since $T_V(s)$ maps $\mathscr{R}$ into $\mathscr{R}$ by composition, each member of the one-parameter family of maps

$$\Phi_V(s) = T_V(s) \circ \Phi \tag{77}$$

is a regular map and (76) gives

$$A[\Phi_V(s)] = \int_{M_4} \Phi_V(s)^*(L\mu) = \int_{M_4} \Phi^* \circ T_V(s)^*(L\mu)$$

$$= \int_{M_4} \Phi^* \exp(s£_V)(L\mu)$$

since $T_V(s)^*\alpha = \exp(s£_V)\alpha$ for any $\alpha \in \Lambda(K)$. Thus, the *finite variation* of the action $A[\Phi]$ is

$$\Delta_V(s)A[\Phi] = \int_{M_4} \Phi^*(\exp(s£_V) - 1)(L\mu) \tag{78}$$

and the infinitesimal variation $\delta_V A[\Phi] = \lim_{s\to 0}(s^{-1}\Delta_V(s)A[\Phi])$ has the evaluation

$$\delta_V A[\Phi] = \int_{M_4} \Phi^* £_V(L\mu) \tag{79}$$

Annihilation of the infinitesimal variation of the action for all $V \in TC(K)$ that do not deform the independent variables x^i (i.e., $V\langle x^i\rangle = 0$) gives the Euler–Lagrange field equations for the ϕ^α's, while the general variation process with $V\langle x^i\rangle \neq 0$ leads to results in the calculus of variations in the large and to the well-known transversality conditions.

This context provides a particularly simple setting for study of symmetries of the action. Let $N_1(L)$ be the collection of all Noetherian vector fields of the first kind (Edelen, 1980):

$$N_1(L) = \{V \in TC(K) | £_V(L\mu) \equiv 0 \bmod C\} \tag{80}$$

Since any regular Φ is such that Φ^* annihilates the contact ideal C, (78) and (80) give the global invariance

$$A[\Phi_V(s)] = A[\Phi] \tag{81}$$

for all $V \in N_1(L)$. Now, $TC(K)$ forms an infinite dimensional Lie subalgebra of $T(K)$ and hence $N_1(K)$ forms a Lie subalgebra of $T(K)$ since $£_{[U,V]}(L\mu) = (£_U£_V - £_V£_U)(L\mu) \equiv 0 \bmod C$. Further, the Lie algebra $N_1(K)$ is almost invariable of finite dimension, say k, in which case there are k independent current $(n-1)$ forms that are conserved for any Φ that renders $A[\Phi]$ stationary in value (Noether, 1918; Edelen, 1980; Edelen, to be published).

The Lie group $N_1(K)$, or a Lie subgroup thereof, provides the obvious candidate for the group G_r considered in previous sections. We therefore consider the case where $G_r \subset N_1(L)$, in which case we have $V_a \in N_1(L)$, $1 \leqslant a \leqslant r$. It is then a simple matter to construct the r-parameter family of maps

$$\Phi(\bar{u}^a) = T(\bar{u}^a) \circ \Phi \tag{82}$$

where the $\bar{u}^a$'s are "constant" canonical parameters of G_r and $T(\bar{u}^a)$ is generated by the flow associated with $V = \bar{u}^a V_a$. In this event, (81) gives the global G_r invariance

$$A[\Phi(\bar{u}^a)] = A[\Phi] \tag{83}$$

where

$$A[\Phi(\bar{u}^a)] = \int_{M_4} \Phi^* \exp(\bar{u}^a \pounds_a)(L\mu) \tag{84}$$

Now,

$$`L`\mu = (\exp(\bar{u}^a \pounds_a) L) \exp(\bar{u}^a \pounds_a)\mu = \exp(\bar{u}^a \pounds_a)(L\mu) = `(L\mu)$$

while $\pounds_a(L\mu) \equiv 0 \bmod C$ gives

$$`L`\mu = `(L\mu) \equiv L\mu \bmod C \tag{85}$$

Thus, since Φ^* annihilates C, we have the G_r invariance

$$`\Phi^*(L\mu) = \Phi^* `(L\mu) = \Phi^*(L\mu) \tag{86}$$

where $`\Phi = T(\bar{u}^a) \circ \Phi$, and

$$`\Phi^* C^\alpha = \Phi^* `C^\alpha = 0 \tag{87}$$

because $\pounds_a C^\alpha = \partial_\beta(V_a \lrcorner C^\alpha) C^\beta$ [i.e., $\pounds_a C \subset C$, see Edelen (1980), Chap. 5] and Φ is regular.

7. THE MINIMAL REPLACEMENT CONSTRUCT

Many gauge theories have been based upon cases in which the group G_r is an *internal* symmetry group of the physical state variables; that is, the action of G_r leaves the manifold M_4 of space-time invariant while changing the ϕ's and the y's. This restriction is clearly not essential (see Kikkawa et al., 1983), for gauge theory constructs rest on the fact that the group G_r is an invariance group of the action $A[\Phi]$ of the physical system under consideration. We therefore take up the general case here and later specialize to the important situation in which G_r is the ten-parameter Poincaré group.

The considerations of the last section were based on the supposition that the canonical parameters $\{\bar{u}^a|1 \leqslant a \leqslant r\}$ were constants. The action of the group G_r on K was therefore global. Thus, the action integral $A[\Phi]$ is invariant under the global action of any $G_r \subset N_1(L)$ for all regular maps Φ. On the other hand, gauge theory arises by allowing different elements of G_r to act at different points of M_4 while preserving the invariance of the action. There are clearly two parts to this problem. The first is to allow the canonical parameters to vary from one point to another over M_4, and the second is to retain the invariance of the action $A[\Phi]$ under the resulting local action of the group G_r.

The simplest way of accomplishing these tasks is to lift considerations from K to the larger space $\mathscr{G} = G_r \times K$ with local coordinates $\{u^a; x^i, q^\alpha, y^\alpha\}$; that is,

$$\{z^A|1 \leqslant A \leqslant 4+5m\} = \{x^i, q^\alpha, y_i^\alpha|1 \leqslant i \leqslant 4, 1 \leqslant \alpha \leqslant m\}$$

$n = 4$ and $N = 5m$. For this general setting, the group space coordinates $\{u^a\}$ are independent quantities that may vary in any way we please. Once things have been analyzed in $\mathscr{G}$, we will be able to consider sections $S: M_4 \to G_r$ without difficulty. There is actually no real choice in the matter, for $S^*(\exp(u^a £_a)\alpha)$ is quite different from $\exp(S^*(u^a)£_a)\alpha$ for $\alpha \in \Lambda(K)$. Put differently, position dependent action of G_r means that different elements of G_r act at different positions, that is, $S^*(\exp(u^a £_a)\alpha)$, not $\exp(S^*(u^a)£_a)\alpha$. In fact, $\exp(S^*(u^a)£_a)z^B$ will belong to G_r only if $S: M_4 \to G_r$ defines a G_r-constant section $\{u^a = k^a|1 \leqslant a \leqslant r\}$.

Let $\Gamma = W^a £_a$ be a Lie connection for the group $G_r \subset N_1(L)$ and assume that the W^a's have the form

$$W^a = W_b^a \, \bar{d}u^b \tag{88}$$

with $£_a\langle W_c^b\rangle = 0$. If $\alpha|_k$ denotes the restriction of any exterior form on $\mathscr{G}$ to a constant section of G_r (i.e., $u^a = \bar{u}^a = \text{const}, 1 \leqslant a \leqslant r$), we have $W^a|_k = 0$ and hence

$$(D\alpha)|_k = d(\alpha|_k), (\bar{d}\alpha)|_k = d(\alpha|_k) \tag{89}$$

This, however, is exactly the case in which the group G_r acts globally on K.

All of the results of the previous section thus lift directly to G_r-constant sections of $\mathscr{G}$.

In order to remove the restriction to G_r-constant sections of $\mathscr{G}$ we note that

$$`(dz^A) = \exp(u^a \pounds_a)\, dz^A \neq \bar{d}\big(\exp(u^a \pounds_a) z^A\big) = \bar{d}`z^A$$

because the u's can change, but

$$`(Dz^A) = \exp(u^a \pounds_a)\, Dz^A = `D\big(\exp(u^a \pounds_a) z^A\big) = `D`z^A$$

by (18). Further, (89) shows that Dz^A restricted to any G_r-constant section of $\mathscr{G}$ agrees with dz^A. Thus, if we simply replace the exterior derivative by the G_r-covariant exterior derivative in all statements in Section 6, these statements become G_r-covariant statements on $\mathscr{G}$.

Let the replacement operator

$$\mathscr{M} : \bar{d} \mapsto D \tag{90}$$

be defined by

$$\mathscr{M}(\alpha + \beta) = \mathscr{M}\alpha + \mathscr{M}\beta, \qquad \mathscr{M}(\alpha \wedge \beta) = (\mathscr{M}\alpha) \wedge (\mathscr{M}\beta),$$

$$\mathscr{M}(\bar{d}z^A) = Dz^A, \qquad \mathscr{M}(\bar{d}u^a) = Du^a = \bar{d}u^a, \qquad \mathscr{M}f = f \;\forall f \in \Lambda^0(\mathscr{G})$$

Thus, since $\Lambda(\mathscr{G})$ is a module over $\Lambda^0(\mathscr{G})$ that is generated from the basis $(1, \bar{d}z^A, \bar{d}u^b)$, $\mathscr{M}$ is well defined on $\Lambda(\mathscr{G})$. Some care must be exercised here, for $\mathscr{M}(d\alpha) \neq D(\mathscr{M}\alpha)$. Simply observe that $\mathscr{M}(d(z^A\, dz^B)) = \mathscr{M}(dz^A \wedge dz^B) = Dz^A \wedge Dz^B$, while $D\mathscr{M}(z^A\, dz^B) = D(z^A\, Dz^B) = Dz^A \wedge Dz^B + z^A DDz^B$ and $DDz^B = \theta^a \pounds_a z^B \neq 0$. On the other hand,

$$\mathscr{M}(\bar{d}f) = \frac{\partial f}{\partial z^A} Dz^A + \frac{\partial f}{\partial u^a} \bar{d}u^a$$

for any $f \in \Lambda^0(\mathscr{G})$. We thus have

$$\mathscr{M}C^\alpha = Dq^\alpha - y_i^\alpha\, Dx^i = C^\alpha + W^a(V_a \lrcorner C^\alpha) \tag{91}$$

$$\mathscr{M}(L\mu) = \mathscr{M}(L)\, Dx^1 \wedge Dx^2 \wedge Dx^3 \wedge Dx^4 \tag{92}$$

while (88) shows that

$$(\mathscr{M}C^\alpha)|_k = C^\alpha, \qquad (\mathscr{M}(L\mu))|_k = L\mu \tag{93}$$

Further, $`(D\alpha) = \exp(u^a \pounds_a)\, D\alpha = `D(\exp(u^a \pounds_a)\alpha) = `D`\alpha$, and $Du^a = \bar{d}u^a$,

${}^{`}u^a = u^a$ because $£_b u^a = 0$. We therefore have ${}^{`}(\mathscr{M}\beta) = \exp(u^a £_a)\mathscr{M}\beta = \mathscr{M}(\exp(u^a £_a)\beta) = \mathscr{M}({}^{`}\beta)$ where $\mathscr{M}(\bar{d}\,{}^{`}z^A) = {}^{`}D\,{}^{`}z^A$:

$$ {}^{`}(\mathscr{M}\beta) = \mathscr{M}({}^{`}\beta)\ \forall \beta \in \Lambda(\mathscr{G}) \tag{94} $$

Accordingly,

$$ {}^{`}(\mathscr{M}(L\mu)) = \mathscr{M}({}^{`}(L\mu)) \tag{95} $$

under the action of the group G_r.

There is quite a bit more here, however, for $(\mathscr{M}(L\mu))|_k = L\mu$ and also $(\mathscr{M}(L\mu)+\eta)|_k = L\mu$ for any $\eta \in \Lambda^4(\mathscr{G})$ that vanishes on G_r-constant sections of $\mathscr{G}$. Further, (95) gives

$$ {}^{`}(\mathscr{M}(L\mu)+\eta) = \mathscr{M}({}^{`}(L\mu)) + \eta \tag{96} $$

provided η is a G_r-invariant 4-form on $\mathscr{G}$ ($\exp(u^a £_a)\eta = \eta$). Thus, *the transition*

$$ L\mu \mapsto \mathscr{M}(L\mu) + \eta \tag{97} $$

for any $\eta \in \Lambda^4(\mathscr{G})$ *such that*

$$ \eta|_k = 0, \qquad {}^{`}\eta = \exp(u^a £_a)\eta = \eta \tag{98} $$

lifts $L\mu \in \Lambda^4(K)$ *up to an element of* $\Lambda^4(\mathscr{G})$ *for which*

$$ (\mathscr{M}(L\mu)+\eta)|_k = L\mu \tag{99} $$

$$ {}^{`}(\mathscr{M}(L\mu)+\eta) = \mathscr{M}({}^{`}(L\mu)) + \eta \tag{100} $$

The partial transition $L\mu \mapsto \mathscr{M}(L\mu)$ will turn out to be the Yang–Mills *minimal replacement*, while $\mathscr{M}(L\mu) \mapsto \mathscr{M}(L\mu)+\eta$ is the basis for the Yang–Mills *minimal coupling* construct.

The construct arrived at in this way is more general than actually needed, for we are interested only in what happens when the cononical parameters $\{u^a | 1 \leqslant a \leqslant r\}$ vary over the space-time manifold M_4. It is therefore necessary to cut things down by introducing mappings

$$ S: M_4 \to G_r | u^a = s^a(x^j) \tag{101} $$

When S acts, (88) gives

$$S^*W^a = (S^*W_b^a)\frac{\partial s^b}{\partial x^i}\,dx^i$$

and hence we may write

$$S^*W^a = W_i^a(x^j)\,dx^i \tag{102}$$

where $\{W_i^a(x^j)|1 \leqslant a \leqslant r,\ 1 \leqslant i \leqslant 4\}$ is taken to be a system of $4r$ *new fields* that compensate for the local space-time action $u^a = s^a(x^j)$ of the group G_r. Now, $S^*\bar{d}\alpha = dS^*\alpha$ and hence

$$S^*D\alpha = dS^*\alpha + W_i^a\,dx^i \wedge S^*£_a\alpha \tag{103}$$

If α depends on the u^a's in any way, $S^*£_a\alpha \neq £_aS^*\alpha$. On the other hand, if $\beta \in \Lambda(K)$ then $S^*\beta = \beta$ and we have

$$S^*D\beta = d\beta + W_i^a\,dx^i \wedge £_a\beta \tag{104}$$

which we will simply write as $D^*\beta$ for $\beta \in \Lambda(K)$ from now on. The G_r-covariant exterior derivative D^* thus induces the G_r-covariant derivative D_i^*, where

$$D_i^*\gamma = \partial_i\gamma + W_i^a£_a\gamma$$

for any linear geometric object field γ on M_4.

A combination of the two operations S^* and $\mathscr{M}$ gives what is usually called the *minimal replacement*

$$\mathscr{M}^* = S^*\mathscr{M} \tag{105}$$

In particular, we have

$$\mathscr{M}^*\,dx^i = D^*x^i = dx^i + W_j^a\,dx^j£_ax^i = \left(\delta_j^i + W_j^a£_ax^i\right)dx^j \tag{106}$$

$$\mathscr{M}^*dq^\alpha = D^*q^\alpha = dq^\alpha + W_j^a£_aq^\alpha\,dx^j \tag{107}$$

and hence (91) and (92) give

$$\mathscr{M}^*C^\alpha = C^\alpha + (V_a \lrcorner C^\alpha)W_i^a\,dx^i \tag{108}$$

$$\mathscr{M}^*(L\mu) = \mathscr{M}(L)\det\left(\delta_j^i + W_j^a£_ax^i\right)\mu \tag{109}$$

The $4r$ quantities $\{W_j^a(x^i)|1 \leqslant a \leqslant r,\ 1 \leqslant j \leqslant 4\}$ constitute a system of new fields that compensate for the space-time dependence of the action of the group G_r. In this vein, it must be clearly noted that we have gone from the system of $r + r^2$ quantities $\{u^a, W_b^a\}$ to the system of $4r$ quantities $\{W_i^a\}$ since the individual u^a's become lost among the other x^i dependences once the map $S: M_4 \to G_r$ has been effected. This, however, is the standard situation in gauge theory, for a specific mapping $S: M_4 \to G_r$ is not obtained, only the compensating fields $\{W_i^a(x)\}$. Accordingly, we must adjoin the $4r$ quantities $\{W_i^a\}$ to the list $\{q^\alpha, y_i^\alpha\}$ as a system of new state variables.

There is now an important question that must be resolved; namely, what is the image of a quantity $S^*\alpha$ under the action of the group G_r? The considerations given at the beginning of this section concerning the nature of the map S^* show that *the action of G_r must be computed in $\mathscr{G}$ and only afterward cut down by sectioning with S*. This means that the image of $S^*\alpha$ can only be defined by

$$ {}^{\backprime}(S^*\alpha) = S^*(\exp(u^a \pounds_a)\alpha) = S^*({}^{\backprime}\alpha) \tag{110} $$

Thus, since $\mathscr{M}^* = S^*\mathscr{M}$, (110) and (94) give ${}^{\backprime}(\mathscr{M}^*\alpha) = {}^{\backprime}(S^*\mathscr{M}\alpha) = S^*({}^{\backprime}(\mathscr{M}\alpha)) = S^*\mathscr{M}({}^{\backprime}\alpha)$; that is,

$$ {}^{\backprime}(\mathscr{M}^*\alpha) = \mathscr{M}^*({}^{\backprime}\alpha) \tag{111} $$

In view of the transition $C^\alpha \to \mathscr{M}^*C^\alpha$, the induced transition of the contact ideal C is

$$ C \to C^* = \mathscr{M}^*C = I\{\mathscr{M}^*C^1, \ldots, \mathscr{M}^*C^m\} \tag{112} $$

Now, $G_r \subset N_1(L)$ so that (85) holds. We thus have

$$ \mathscr{M}^*({}^{\backprime}(L\mu)) \equiv \mathscr{M}^*(L\mu) \bmod C^* $$

and hence (111) gives

$$ {}^{\backprime}\mathscr{M}^*(L\mu) \equiv \mathscr{M}^*(L\mu) \bmod C^* \tag{113} $$

For $S: M_4 \to G_r$, the transition (97) becomes

$$ L\mu \to \mathscr{M}^*(L\mu) + S^*\eta \tag{114} $$

The new action integral is thus given by

$$ \bar{A}[\Phi] = \int_{M_4} \Phi^*(\mathscr{M}^*(L\mu) + S^*\eta) \tag{115} $$

for any regular map

$$\Phi: M_4 \to K \times \mathbb{R}^{4r} | x^i = x^i, \qquad q^\alpha = \phi^\alpha(x^j), \qquad W_i^a = W_i^a(x^j)$$

$$\Phi^* \mathcal{M}^* \mu \neq 0, \qquad \Phi^* C^* = 0 \tag{116}$$

(recall that minimal replacement induces the transition $\mu \to \mathcal{M}^*\mu$ and that the quantities $\{W_i^a\}$ are to be included as new field variables). We now simply observe that (98) and (110) give $`(S^*\eta) = S^*\eta$ and hence

$$`(\mathcal{M}^*(L\mu) + S^*\eta) \equiv \mathcal{M}^*(L\mu) + S^*\eta \bmod C^* \tag{117}$$

by (113). Accordingly, (115)–(117) show that *the new action integral*, $\bar{A}[\Phi]$, *is invariant under the local action of the Lie group* G_r.

It should be noted that we started in K where each V_a that generates G_r is an isovector of the contact ideal,

$$£_b C^\alpha = A^\alpha_{b\beta} C^\beta \tag{118}$$

for which $£_b(L\mu) \equiv 0 \bmod C$. Under minimal replacement, C^α is replaced by $\mathcal{M}^* C^\alpha = C^\alpha + W^a(V_a \lrcorner C^\alpha)$. Thus, when (118) is used, we have

$$£_b \mathcal{M}^* C^\alpha = (V_e \lrcorner C^\alpha)(£_b W^e + W^a C^e_{ba}) \bmod C^* \tag{119}$$

The generators of G_r *fail to be isovectors of* C^*. Thus, although the new action $\bar{A}[\Phi]$ is G_r invariant, the contact forms are only G_r covariant:

$$`(\mathcal{M}^* C^\alpha) = \mathcal{M}^*(`C^\alpha) = `D`q^\alpha - `y_i^\alpha `D`x^i \tag{120}$$

What this means is that the r conserved currents that arise from global action of G_r go over into r balanced currents that are integrability conditions on the field equations for the compensating fields of the local action of G_r, as we shall see in the following section.

8. VARIATIONS AND THE FIELD EQUATIONS

The problem at hand is that of obtaining the governing Euler–Lagrange field equations. These obtain from rendering the action

$$\bar{A}[\Phi] = \int_{M_4} \Phi^*(\mathcal{M}^*(L\mu) + S^*\eta) \tag{121}$$

stationary in value subject to the constraints

$$\Phi^* C^* = 0, \qquad \Phi^* \mathscr{M}^* \mu \neq 0 \tag{122}$$

where

$$\begin{gathered} \Phi : M_4 \to K \times \mathbb{R}^{4r} | x^i = x^i, \qquad q^\alpha = \phi^\alpha(x^j), \qquad W_i^a = W_i^a(x^j) \\ \Phi^* \mu \neq 0, \qquad \Phi^* \mathscr{M}^* C^\alpha = 0 \end{gathered} \tag{123}$$

and η is a G_r-invariant 4-form on $\mathscr{G}$ that vanishes on G_r-constant sections of G_r. Now,

$$\mathscr{M}^* \, dx^i = D^* x^i = T_j^i \, dx^j \tag{124}$$

where

$$T_j^i = \delta_j^i + W_j^a \pounds_a x^i \tag{125}$$

and hence

$$\mathscr{M}^* \mu = \det\left(T_j^i\right) \mu \tag{126}$$

We therefore have $\mathscr{M}^*(L\mu) = \mathscr{M}^*(L) \det(T_j^i)\mu$. Now, $L \in \Lambda^0(K)$, $L = L(x^i, q^\alpha, y_i^\alpha)$, and hence $\mathscr{M}^*(L) = L$; that is,

$$\mathscr{M}^*(L\mu) = L \det\left(T_j^i\right) \mu \tag{127}$$

The reader accustomed to the standard minimal replacement construct might expect to see the y_i^α's change in L. This is not the case here, for the y_i^α's are independent quantities in the space K. We shall see, however, that $\Phi^* y_i^\alpha$ will be drastically different as a consequence of satisfaction of the constraints $\Phi^* \mathscr{M}^* C^\alpha = 0$ rather than $\Phi^* C^\alpha = 0$. Minimal replacement does have its expected effect on the derivatives of the field variables, but these effects come about only after application of Φ^*.

The exact nature of the 4-form η, and hence $S^*\eta$, is somewhat arbitrary at this point, although $S^*\eta$ must account for the presence of G_r-curvature terms. Accordingly, we shall deal with the problem in the general form

$$\mathscr{M}^*(L\mu) + S^*\eta = \bar{L}\mu \tag{128}$$

with

$$\bar{L} = \bar{L}\left(x^j, q^\alpha, y_i^\alpha, W_i^a, \Theta_{ij}^a\right) \tag{129}$$

Here, we have set

$$S^*\theta^a = \Theta^a = \tfrac{1}{2}\Theta_{ij}^a \, dx^i \wedge dx^j, \qquad \Theta_{ij}^a = -\Theta_{ji}^a$$

$$\Theta^a = dW^a + \tfrac{1}{2}C_{be}^a W^b \wedge W^e, \qquad W^a = W_i^a \, dx^i \tag{130}$$

The easiest way of dealing with this variational problem is to shift directly to the space R with local coordinates (x^i, q^α, W_i^a). A vector field on R has the form

$$U = U^i \partial_i + U^\alpha \partial_\alpha + U_i^a \partial_a^i$$

where we have set $\partial_a^i = \partial / \partial W_i^a$. The classic variational process requires increments of the field variables that are functions on M_4, while the points of M_4 itself are unchanged by the variation process. It is therefore sufficient to our purposes to take $U^i = 0$ and all of the remaining U's to be functions of the x^i's only; that is,

$$U = U^\alpha(x^j)\, \partial_\alpha + U_i^a(x^j)\, \partial_a^i \tag{131}$$

We may then use Lie differentiation with respect to U to compute the variations in y_i^α and Θ_{ij}^a that arise from the variations (U^α, U_i^a) in the basic fields (q^α, W_i^a), respectively.

The induced variations in the y_i^α's are obtained through satisfaction of the conditions

$$\pounds_U \mathscr{M}^* C^\alpha = 0 \tag{132}$$

that is, the variations preserve the constraints (122). Now, a combination of (106), (107), (108), and (125) yield

$$\mathscr{M}^* C^\alpha = dq^\alpha + W_i^a \pounds_a q^\alpha \, dx^i - y_i^\alpha T_j^i \, dx^j \tag{133}$$

We therefore have

$$\Phi^* \mathscr{M}^* C^\alpha = \left(\partial_j \phi^\alpha + W_j^a \Phi^*(\pounds_a q^\alpha) - \Phi^* y_i^\alpha \Phi^* T_j^i \right) dx^j$$

and hence satisfaction of the constraints (122) demands that

$$\Phi^*(y_i^\alpha)\Phi^*\left(T_j^i\right) = \partial_j\phi^\alpha + W_j^a\Phi^*(\pounds_a q^\alpha)$$

for any map Φ of the form (123). These are the relations that determine the place holders y_i^α in $\overline{L}$ when we come down to actual evaluations in terms of the fields $\{\phi^\alpha(x^j), W_i^a(x^j)\}$.

Noting that (131) gives

$$\pounds_U\pounds_a x^i = U^\beta\partial_\beta\pounds_a x^i, \qquad \pounds_U\pounds_a q^\alpha = U^\beta\partial_\beta\pounds_a q^\alpha$$

(132) and (133) lead to

$$\begin{aligned}(\pounds_U y_i^\alpha)T_j^i &= \partial_j U^\alpha + U_j^a\pounds_a q^\alpha + U^\beta W_j^a\partial_\beta\pounds_a q^\alpha \\ &\quad - y_i^\alpha\left(U_j^a\pounds_a x^i + U^\beta W_j^a\partial_\beta\pounds_a x^i\right)\end{aligned} \tag{134}$$

It is clear from (126) and $\Phi^*\mathscr{M}^*\mu \neq 0$ that we must require

$$\det\left(T_j^i\right) \neq 0 \tag{135}$$

and hence we may introduce the quantities t_j^i by

$$T_j^i t_k^j = \delta_k^i \tag{136}$$

Thus, (134) yields the desired specific evaluation

$$\begin{aligned}\pounds_U y_i^\alpha = \Big(&\partial_j U^\alpha + U_j^a\pounds_a q^\alpha + U^\beta W_j^a\partial_\beta\pounds_a q^\alpha \\ &- y_k^\alpha\left(U_j^a\pounds_a x^k + U^\beta W_j^a\partial_\beta\pounds_a x^k\right)\Big)t_i^j\end{aligned} \tag{137}$$

Computation of the variations that are induced in Θ_{ij}^a are most easily accomplished by noting that (131) yields

$$\pounds_U W^a = U_i^a\, dx^i \tag{138}$$

Accordingly, (102) gives

$$\pounds_U\Theta^a = d\pounds_U W^a + \pounds_U W^b \wedge C_{bc}^a W^c \tag{139}$$

Further expansion is unnecessary, as we shall see presently.

We now have all of the results needed to proceed with the final calculations. Since $\pounds_U(\bar{L}\mu) = (\pounds_U \bar{L})\mu$ because $\pounds_U \mu = 0$ (recall that $U^i = 0$), we need only compute $\pounds_U \bar{L}$. Thus, introducing the notation

$$L_\alpha = \partial \bar{L}/\partial q^\alpha, \qquad L_\alpha^i = \partial \bar{L}/\partial y_i^\alpha \tag{140}$$

$$G_a^{ij} = \partial \bar{L}/\partial \Theta_{ij}^a, \qquad \sigma_a^i = \partial \bar{L}/\partial W_i^a|_{\Theta^b} \tag{150}$$

we have

$$\pounds_U(\bar{L}\mu) = \left(L_\alpha U^\alpha + L_\alpha^i \pounds_U y_i^\alpha + \sigma_a^i U_i^a + G_a^{ij} \pounds_U \Theta_{ij}^a \right)\mu \tag{151}$$

It is now simply a matter of substituting (137) and (139) into (151) and then discarding all divergences and/or exact 4-forms in order to obtain the Euler–Lagrange field equations.

The field equations for the q^α's (for the ϕ^α's) come from collecting together all terms that involve the U^α's and their derivatives that appear in (151):

$$\partial_j\left(t_i^j L_\gamma^i U^\gamma \right)\mu + U^\gamma\left\{ L_\gamma - \partial_j\left(t_i^j L_\gamma^i \right) + t_i^j L_\alpha^i W_j^a\left(\partial_\gamma \pounds_a q^\alpha - y_k^\alpha \partial_\gamma \pounds_a x^k \right\}\mu\right.$$

Standard practices of the calculus of variations thus give *the Euler–Lagrange equations for the q^α fields*:

$$\Phi^*\left\{ \partial_j\left(t_i^j L_\alpha^i \right) \right\} = \Phi^*\left\{ L_\alpha + t_i^j L_\beta^i W_j^a\left(\partial_\alpha \pounds_a q^\beta - y_k^\beta \partial_\alpha \pounds_a x^k \right) \right\} \tag{152}$$

The terms in (133) that involve the variations U_i^a in the W^a fields are given by

$$\xi = \left[U_j^a\left\{ \sigma_a^j + t_i^j L_\alpha^i\left(\pounds_a q^\alpha - y_k^\alpha \pounds_a x^k \right) \right\} + G_a^{ij} \pounds_U \Theta_{ij}^a \right]\mu \tag{153}$$

If we set

$$J_a = \left\{ \sigma_a^j + t_i^j L_\alpha^i\left(\pounds_a q^\alpha - y_k^\alpha \pounds_a x^k \right) \right\}\mu_j \in \Lambda^3 \tag{154}$$

$$G_a = \tfrac{1}{2} G_a^{ij} \mu_{ij} \in \Lambda^2 \tag{155}$$

where

$$\mu_{ij} = \partial_i \lrcorner \mu_j, \qquad dx^k \wedge \mu_{ij} = \delta_i^k \mu_j - \delta_j^k \mu_i \tag{156}$$

then (130), (138), and (153)–(156) give the particularly simple evaluation

$$\xi = \pounds_U W^a \wedge J_a - 2\pounds_U \Theta^a \wedge G_a \tag{157}$$

When (139) is used, an elementary rearrangement gives

$$\pounds_U \Theta^a \wedge G_a = d(\pounds_U W^a \wedge G_a) + \pounds_U W^a \wedge \left(dG_a + C^b_{ac} W^c \wedge G_b\right)$$

We therefore have

$$\xi = \pounds_U W^a \wedge \left\{ J_a - 2\,dG_a - 2C^b_{ac} W^c \wedge G_b \right\} - 2d(\pounds_U W^a \wedge G_a) \tag{158}$$

Standard practices of the calculus of variations thus give *the Euler – Lagrange equations for the W_i^a fields*:

$$\Phi^*\left\{ dG_a + C^b_{ac} W^c \wedge G_b \right\} = \tfrac{1}{2}\Phi^* J_a \tag{159}$$

The field equations (159) obviously entail integrability conditions. If we write (159) in the equivalent form ($G_a^* = \Phi^* G_a$, $J_a^* = \Phi^* J_a$)

$$dG_a^* = \tfrac{1}{2} J_a^* - C^b_{ac} W^c \wedge G_b^* \tag{160}$$

then exterior differentiation gives

$$\tfrac{1}{2}\, dJ_a^* = C^b_{ac}\left(dW^c \wedge G_b^* - W^c \wedge dG_b^* \right) \tag{161}$$

When (160) and $\Theta^c = dW^c + \frac{1}{2} C^c_{ef} W^e \wedge W^f$ are used to eliminate dG_b^* and dW^c from the right-hand side of (161) and the Jacobi identity is applied, *the integrability conditions for the W_i^a-field equations are*

$$dJ_a^* + C^b_{ac} W^c \wedge J_b^* = 2C^b_{ac} \Theta^c \wedge G_b^* \tag{162}$$

If the dependence of the Lagrangian $\bar{L}$ on W_i^a and Θ^a_{ij} is such that

$$C^b_{ac} \Theta^c \wedge G_b^* = 0 \tag{163}$$

which would appear to be the case as a consequence of G_r invariance (Rund, 1982), we obtain the G_r-covariant current conservation laws

$$dJ_a^* + C^b_{ac} W^c \wedge J_b^* = 0 \tag{164}$$

We saw at the end of Section 7 that

$$£_b \mathscr{M} {}^*C^\alpha \equiv (V_e \lrcorner C^\alpha)(£_b W^e + W^a C^e_{ba})$$

so that G_r is not generated by isovectors of the ideal C^*. The r exact 4-forms (conservation laws) that are implied by $G_r \subset N_1(K)$ would appear to have been lost. What has actually happened is that the minimal replacement construct carries these conservation laws over into the system of r integrability conditions (162) or (164). In the case of (164), we recognize a system of G_r-covariant conservation laws that replace the usual ones on G_r-constant sections.

9. GAUGE THEORY FOR THE POINCARE GROUP

Most variational principles of current interest in physics (Bernstein, 1974; Weinberg, 1974; Sirlin, 1978) are manifestly invariant under the ten-parameter Poincaré group, $P_{10}(\mathbb{R}) = L(4,\mathbb{R}) \triangleright T(4)$, where $L(4,\mathbb{R})$ is the Lorentz group, $T(4)$ is the four-parameter translation group, and $\triangleright$ denotes the semidirect product. In addition, the flat space-time manifold M_4 carries a Lorentz structure

$$ds^2 = h_{ij}\, dx^i \otimes dx^j, \qquad ((h_{ij})) = \mathrm{diag}(1,1,1,-1) \tag{165}$$

for which $P_{10}(\mathbb{R})$ is the maximal group of isometries.

In view of the semidirect product structure of P_{10}, it is natural that we decompose the canonical parameters $\{u^a | 1 \leqslant a \leqslant 10\}$ into two sets by

$$\{u^a | 1 \leqslant a \leqslant 10\} = \{u^r; u^i | 1 \leqslant r \leqslant 6, 1 \leqslant i \leqslant 4\}$$

If $\{v_a | 1 \leqslant a \leqslant 10\}$ is a basis for the Lie algebra of P_{10} realized as a group of automorphisms of M_4, we have

$$v = u^a v_a = u^r l^i_{rj} x^j \partial_i + u^i \partial_i \tag{166}$$

where the $\{l^i_{rj}\}$ is a basis for the matrix Lie algebra of $L(4,\mathbb{R})$;

$$h_{ik} l^k_{rj} + h_{jk} l^k_{ri} = 0 \tag{167}$$

Now,

$$[v_a, v_b] = C^e_{ab} v_e \tag{168}$$

where the C's are the structure constants of P_{10}, while (166) shows that $[\partial_i, \partial_j] = 0$. The Cartan–Killing form $\{C_{ab}\}$ thus has rank equal to six. The 6-by-6 form $\{C_{rs} | 1 \leqslant r, s \leqslant 6\}$ is nonsingular if we identify the first six u^a's with the six u^r's. We therefore set

$$u^a = \delta_r^a u^r + \delta_{6+i}^a u^i \tag{169}$$

The statement that the action is invariant under P_{10} means that P_{10} must be lifted to an isomorphic global group G_{10} of Noetherian vector fields on kinematic space K. Now, K is a $(4+5m)$-dimensional space with local coordinates $\{x^i, q^\alpha, y_i^\alpha\}$, so we must say something about how the state variables $\{q^\alpha\}$ behave when M_4 is subjected to the action of P_{10}. It is reasonable to assume that $\{\Phi^* q^\alpha\}$ transform under the global action of P_{10} as linear differential geometric object fields (as combinations of scalors, vectors, tensors, etc.). As such, the global translation part, $T(4)$, of P_{10} will have no effect and we may write

$$u^a V_a = u^i \partial_i + u^r l_{rj}^i x^j \partial_i + u^r M_{r\beta}^\alpha q^\beta \partial_\alpha + u^a Z_k (V_a \lrcorner C^\alpha) \partial_\alpha^i \tag{170}$$

Here, $\partial_\alpha = \partial / \partial q^\alpha$, $\partial_\alpha^i = \partial / \partial y_i^\alpha$, and the M's are constants that are determined by the transformation properties of the q^α's and are such that

$$[V_a, V_b] = C_{ab}^e V_e \tag{171}$$

that is,

$$M_{r\beta}^\alpha M_{s\alpha}^\gamma - M_{s\beta}^\alpha M_{r\alpha}^\gamma = c_{rs}^t M_{t\beta}^\gamma, \qquad 1 \leqslant r, s, t \leqslant 6 \tag{172}$$

Here, the lower-case C's are the structure constants of $L(4, \mathbb{R})$. For example, if four of the q's, say, $\{T^i\}$, constitute the components of a vector field on M_4 when pulled back by any regular map $\Phi: M_4 \to K$, we will have

$$`T^i = T^j \partial `x^i / \partial x^j = T^j \left(\delta_j^i + u^r l_{rj}^i + o(u^r) \right)$$

$$= T^i + u^r l_{rj}^i T^j + o(u^r)$$

Hence, the corresponding terms in (170) will be given by $u^r l_{rj}^i T^j (\partial / \partial T^i)$. These clearly satisfy (172).

The minimal replacement construct for P_{10} may now be obtained without further ado; simply apply the results obtained in Section 7. To this end, we set

$$W^a = \delta_r^a W^r + \delta_{6+i}^a W^i \tag{173}$$

in conformity with (169), where

$$W^r = W_i^r\, dx^i, \qquad 1 \leqslant r \leqslant 6 \tag{174}$$

are the compensating fields for $L(4,\mathbb{R})$ and

$$W^i = W_j^i\, dx^j, \qquad 1 \leqslant i \leqslant 4 \tag{175}$$

are the compensating fields for $T(4)$. Thus, since $D^*x^i = dx^i + W^a £_a x^i$, $D^*q^\alpha = dq^\alpha + W^a £_a q^\alpha$, we have

$$D^*x^i = \left(\delta_j^i + W_j^i + W_j^r l_{rk}^i x^k\right) dx^j \tag{176}$$

$$D^*q^\alpha = dq^\alpha + W_j^r M_{r\beta}^\alpha q^\beta\, dx^j \tag{177}$$

Accordingly, (133) and (127) yield

$$\mathscr{M}^*C^\alpha = dq^\alpha + W_j^r \dot{M}_{r\beta}^\alpha q^\beta\, dx^j - y_i^\alpha\left(\delta_j^i + W_j^i + W_j^r l_{rk}^i x^k\right) dx^j \tag{178}$$

$$\mathscr{M}^*(L\mu) = L\det\left(\delta_j^i + W_j^i + W_j^r l_{rk}^i x^k\right)\mu \tag{179}$$

An interesting and characteristic result now obtains. If we apply the minimal replacement operator to the Lorentz structure (165), we have

$$dS^2 = \mathscr{M}^*(ds^2) = T_k^i h_{ij} T_l^j\, dx^k \otimes dx^l \tag{180}$$

where we have set [see (126)]

$$T_k^i = \delta_k^i + W_k^i + W_k^r l_{rj}^i x^j \tag{181}$$

Accordingly, we may write

$$dS^2 = g_{ij}\, dx^i \otimes dx^j \tag{182}$$

where

$$g_{ij} = T_i^k h_{kl} T_j^l = g_{ji} \tag{183}$$

Minimal replacement may thus be viewed as a construct that replaces the Lorentz structure ds^2 on M_4 by the more complicated pseudo-Riemannian structure dS^2 through the transition process $h_{ij} \mapsto g_{ij}$. It is assumed that the minimal replacement construct is *regular* in the sense that

$$\det\left(T_j^i\right) \neq 0 \tag{184}$$

which is clearly necessary in view of (179). (183) then shows that g_{ij} and h_{ij} both have the same signature, namely, 2, and that dS^2 defines a proper pseudo-Riemannian structure on M_4. Further, it follows directly from (183) that

$$\det(g_{ij}) = \det(h_{ij})\det\left(T_j^i\right)^2 = -\det\left(T_j^i\right)^2 \tag{185}$$

Thus, if we set

$$g = \det(g_{ij}) \tag{186}$$

which is necessarily negative, (185) gives

$$\det\left(T_j^i\right) = (-g)^{1/2} \tag{187}$$

Accordingly, when (181) is used, (179) may be rewritten in the equivalent form

$$\mathscr{M}^*(L\mu) = L(-g)^{1/2}\mu, \tag{188}$$

while (178) becomes

$$\mathscr{M}^*C^\alpha = dq^\alpha + \left(W_j^r M_{r\beta}^\alpha q^\beta - y_i^\alpha T_j^i\right) dx^j \tag{189}$$

The form $L(-g)^{1/2}\mu$ given by (188) is immediately recognized as the standard form of an action 4-form on a pseudo-Riemannian space-time with fundamental metric form $g_{ij}\, dx^i \otimes dx^j$.

The total action functional in this new context is given by

$$\bar{A}[\Phi] = \int_{M_4} \Phi^*\left(L(-g)^{1/2}\mu + S^*\eta\right) \tag{190}$$

where η is a G_{10}-invariant 4-form on G_{10} and

$$\begin{gathered} \Phi: M_4 \to K \times \mathbb{R}^{40} | x^i = x^i, \qquad q^\alpha = \phi^\alpha(x^j), \qquad W_i^a = W_i^a(x^j) \\ \Phi^*\mu \neq 0, \qquad \Phi^*\mathscr{M}^*C^\alpha = 0 \end{gathered} \tag{191}$$

see (121) and (123).

In order that we may determine possible forms for η, we first calculate the forms of the curvature 2-forms, Θ^a, for G_{10}. Since $\Theta^a = dW^a + \frac{1}{2}C^a_{bc}W^bW^c$, (169) induces the decomposition

$$\Theta^a = \delta^a_r\Theta^r + \delta^a_{6+i}\Theta^i \tag{192}$$

Noting that G_{10} and P_{10} have the same structure constants, we have

$$C^i_{rs} = 0, \qquad C^a_{ij} = 0, \qquad C^s_{ri} = 0 \tag{193}$$

with $1 \leqslant i,\ j \leqslant 4$, $1 \leqslant r,\ s \leqslant 6$. It is then a simple matter to see that

$$\Theta^r = dW^r + \tfrac{1}{2}C^r_{st}W^s \wedge W^t, \qquad 1 \leqslant r, s, t \leqslant 6 \tag{194}$$

and

$$\Theta^k = dW^k + C^k_{si}W^s \wedge W^i, \qquad 1 \leqslant i, k \leqslant 4, \quad 1 \leqslant s \leqslant 6 \tag{195}$$

The 2-forms $\{\Theta^r | 1 \leqslant r \leqslant 6\}$ are the curvature 2-forms associated with the local action of $L(4,\mathbb{R})$, while $\{\Theta^k | 1 \leqslant k \leqslant 4\}$ are the curvature 2-forms associated with the semidirect product action of $T(4)$, as evidenced by the coupling terms $C^k_{si}W^s \wedge W^i$. Now, $((C_{rs}))$ is a nonsingular 6-by-6 matrix and hence applying S^* shows that we have the G_{10}-invariant scalar

$$\alpha = \Theta^r_{ij}h^{ik}h^{jl}\Theta^s_{kl}C_{rs} \tag{196}$$

where $2\Theta^r = \Theta^r_{ij}\,dx^i \wedge dx^j$. Here, we have used the fact that G_{10} is a group of isometries of M_4 and hence $'h_{ij} = h_{ij}$.

An inspection of (196) shows that α is independent of the curvature coefficients $\{\Theta^k_{ij}\}$ associated with $T(4)$. This is a direct consequence of the fact that G_{10} is *not* a semisimple group and has been a source of certain difficulties in the past; it is necessary that we go back to $\Lambda(\mathscr{G})$ in order to determine other G_{10}-invariant quantities. This problem has been solved for $SO(3) \rhd T(3)$ in Kadić and Edelen (1983) and suggests a possible resolution for G_{10} (see Appendix). Another possibility is afforded by the scalar invariants that can be formed from the Riemannian curvature tensor based on the metric tensor $g_{ij} = T_i^{\ k}h_{kl}T_j^l$. Such invariants have the right properties, for (181) shows that $g_{ij} = h_{ij}$ for G_{10}-constant sections of $\mathscr{G}$ ($W^a = 0$) and the Riemannian curvature tensor formed from h_{ij} vanishes throughout M_4. In any event, even after further invariants are found, there is still the question of selecting an appropriate representation for $S^*\eta$. We leave this aspect of the problem for a future communication and simply take

$$\mathscr{M}^*(L\mu) + S^*\eta = \left(L(-g)^{1/2} + f(\alpha, \ldots)\right)\mu = \bar{L}\mu \tag{197}$$

with

$$\bar{L} = \bar{L}\left(x^j, q^\alpha, y_i^\alpha, W_j^i, W_j^r, \Theta_{jk}^i, \Theta_{jk}^r\right) \tag{198}$$

in conformity with (129). The field equations for the gauge theory of the Poincaré group then follow directly from the results given in Section 8:

$$\Phi^*\partial_j\left(t_i^j L_\alpha^i\right) = \Phi^*\left\{L_\alpha + t_i^j L_\beta^i W_j^a \partial_\alpha £_a q^\beta\right\} \tag{199}$$

$$\left.\begin{aligned} dG_i^* + C_{ic}^b W^c \wedge G_b^* &= \tfrac{1}{2}J_i^*, \qquad 1 \leqslant i \leqslant 4 \\ dG_r^* + C_{rc}^b W^c \wedge G_b^* &= \tfrac{1}{2}J_r^*, \qquad 1 \leqslant r \leqslant 6 \end{aligned}\right\} \tag{200}$$

The reader should compare the results of this section with those reported in Kikkawa et al. (1983) and the references cited therein.

10. OBSERVATIONS

The theory of operator-valued connections has been shown to lead to a simple and direct gauge theory construction for any Lie group G_r of symmetries of a variational principle. The symmetries themselves can be of a general nature. All that is required is that G_r act on the underlying kinematic space K as a Lie group of point transformations. The action of G_r can thus be quite nonlinear, as opposed to previous theories in which G_r has been required to act as a subgroup of the general linear group on the vector space of states. Further, G_r is permitted to act on both the space of states and on the underlying space-time manifold in an indiscriminate manner. An explicit example of this has been given where G_r is the Poincaré group and the action functional is Poincaré invariant.

The generality afforded by the theory of operator-valued connections opens new and possibly fundamental areas of study. An obvious candidate is the construction of gauge theories in which G_r contains the 15-parameter conformal group on flat space-time M_4. The conformal group is the fundamental group of electrodynamics on space-time so it is naturally associated with the action of the electromagnetic field. In addition, the gauge fields that compensate for local deformations of the fundamental form, $£_a h_{ij} = \gamma_a h_{ij}$, provide natural dynamic scale variables that would appear to have fundamental significance. The five integrability conditions (164) in addition to those that arise from P_{10} (G_r-covariant conservation laws) then assume particular significance.

It should be pointed out that a fundamental revision of the theory can be made whereby both the compensating fields $\{W^a\}$ and the group parameter fields $S^*u^\alpha = s^\alpha(x^j)$ are retained. We should then be able to account for any particular gauge field by actually getting hold of the family of specific transformations that map from a G_r-constant section to a space-time-dependent section of group space. This would be of particular importance in the case of the Poincaré group.

APPENDIX. MATRIX REPRESENTATION OF THE POINCARÉ GROUP AND INVARIANTS

Let V_5 be a five-dimensional vector space and consider the affine set of vectors

$$\hat{\mathbf{x}} = [x^1, x^2, x^3, x^4, 1]^T \tag{A1}$$

where $\mathbf{x} = [x^1, x^2, x^3, x^4]^T$ is any position vector in M_4. If $\mathbf{L}$ is a 4-by-4 Lorentz transformation matrix and $\mathbf{t} = [t^1, t^2, t^3, t^4]^T$, then the Poincaré group may be realized as a matrix subgroup of $GL(5)$ consisting of all matrices of the form

$$\mathbf{M} = \begin{pmatrix} \mathbf{L} & \mathbf{t} \\ [0] & 1 \end{pmatrix} \tag{A2}$$

that is,

$$'\hat{x} = \mathbf{M}\hat{\mathbf{x}} = \begin{Bmatrix} \mathbf{Lx} + \mathbf{t} \\ 1 \end{Bmatrix} \tag{A3}$$

Let $\{\mathbf{L}_r | 1 \leqslant r \leqslant 6\}$ be a basis for the matrix Lie algebra of $L(4, \mathbb{R})$, let $\{\mathbf{e}_i | 1 \leqslant i \leqslant 4\}$ generate $T(4)$, and consider

$$\hat{\Gamma} = \begin{pmatrix} W^r\mathbf{L}_r & W^i\mathbf{e}_i \\ [0] & 0 \end{pmatrix} = \begin{pmatrix} \Gamma & \omega \\ [0] & 0 \end{pmatrix} \tag{A4}$$

where Γ may be viewed as the gauge connection for $L(4, \mathbb{R})$. It then follows that (A1) and (A4) serve to reproduce (176) through

$$D^*\hat{\mathbf{x}} = d\hat{\mathbf{x}} + \hat{\Gamma}\hat{\mathbf{x}} = \begin{Bmatrix} d\mathbf{x} + \Gamma\mathbf{x} + \omega \\ 0 \end{Bmatrix} \tag{A5}$$

In like manner, if we set

$$\hat{\Theta} = \begin{pmatrix} \Theta^r \mathbf{L}_r & \Theta^i \mathbf{e}_i \\ [0] & 0 \end{pmatrix} = \begin{pmatrix} \Theta & \Omega \\ [0] & 0 \end{pmatrix} \tag{A6}$$

and note that $D^* D^* \hat{\mathbf{x}}$ means $S^*(D D \hat{\mathbf{x}})$, then

$$D^* D^* \hat{\mathbf{x}} = \hat{\Theta} \hat{\mathbf{x}} \tag{A7}$$

We next note that the transformation laws for $\hat{\Gamma}$ and $\hat{\Theta}$ are

$$`\hat{\Gamma} = \mathbf{M}\hat{\Gamma}\mathbf{M}^{-1} - d\mathbf{M}\,\mathbf{M}^{-1}, \qquad `\hat{\Theta} = \mathbf{M}\hat{\Theta}\mathbf{M}^{-1} \tag{A8}$$

Use of (A2), (A4), and (A6) thus give

$$`\Gamma = \mathbf{L}\Gamma\mathbf{L}^{-1} - d\mathbf{L}\,\mathbf{L}^{-1}$$

$$`\omega = \mathbf{L}\omega - d\mathbf{t} - (\mathbf{L}\Gamma\mathbf{L}^{-1} - d\mathbf{L}\,\mathbf{L}^{-1})\mathbf{t}$$

$$`\Theta = \mathbf{L}\Theta\mathbf{L}^{-1}$$

$$`\Omega = \mathbf{L}\Omega - \mathbf{L}\Theta\mathbf{L}^{-1}\mathbf{t} \tag{A9}$$

The result $`\Theta = \mathbf{L}\Theta\mathbf{L}^{-1}$ leads directly to the invariant given by (196). On the other hand, the occurrence of the translation, $\mathbf{t}$, in the last of (A9) shows that we may not construct an invariant from Ω.

The quantity

$$\hat{\Sigma} = [\Sigma^T, 0]^T = \hat{\Theta}\hat{\mathbf{x}} = \left[(\Theta\mathbf{x} + \Omega)^T, 0\right]^T \tag{A10}$$

has the evaluation $D^* D^* \mathbf{x}$, by (A7), and is thus the Cartan torsion matrix for the differential system constructed from the soldering matrix $D^* \hat{\mathbf{x}}$. Further, (A3), the second of (A8), and (A11) give

$$`\hat{\Sigma} = `\hat{\Theta}`\hat{\mathbf{x}} = \mathbf{M}\hat{\Theta}\mathbf{M}^{-1}\mathbf{M}\hat{\mathbf{x}} = \mathbf{M}\hat{\Theta}\hat{\mathbf{x}} = [\mathbf{L}\Sigma, 0]^T \tag{A11}$$

Accordingly, since $\mathbf{L}^T \mathbf{h} \mathbf{L} = \mathbf{h}$, $\mathbf{h} = ((h_{ij}))$,

$$\beta = \Sigma^T \mathbf{h} \otimes \Sigma \tag{A12}$$

is Poincaré invariant. Thus, if we write [see (A6)]

$$\Sigma^i = \tfrac{1}{2} \Sigma^i_{jk}\, dx^j \wedge dx^k = \Theta^r L^i_{rj} x^j + \Theta^i \tag{A13}$$

we obtain the scalar invariant

$$\beta = \Sigma^i_{kl} h_{ij} \Sigma^j_{mn} h^{km} h^{ln} \tag{A14}$$

This invariant contains the $T(4)$ curvature terms Θ^i.

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